

TOPOLOGICAL QUANTUM CODES FROM SELF-COMPLEMENTARY SELF-DUAL GRAPHS

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In this paper we present two new classes of binary quantum codes with minimum distance of at least three, by self-complementary self-dual orientable embeddings of “voltage graphs” and “Paley graphs in the Galois field $GF(p^r)$ ”, where $p \in \mathbb{P}$ and $r \in \mathbb{Z}^+$. The parameters of two new classes of quantum codes are $[[(2k' + 2)(8k' + 7), 2(8k'^2 + 7k'), d_{min}]]$ and $[[(2k' + 2)(8k' + 9), 2(8k'^2 + 9k' + 1), d_{min}]]$ respectively, where $d_{min} \geq 3$. For these quantum codes, the code rate approaches 1 as k' goes to infinity.

Keywords: quantum codes; embedding; self-complementary; self-dual; voltage graph; Paley graph.

1. Introduction

Quantum error-correcting codes (QEC) plays an important role in the theory of quantum information and computation. A main difficult to realize quantum computation is decoherence of quantum bits due to the interaction between the system and the surrounding environments. The QEC provide an efficient way to overcome decoherence. The first quantum code $[[9, 1, 3]]$ was discovered by Shor [1]. Calderbank et al. [2] have introduced a systematic way for constructing the QEC from classical

error-correcting code. The problem of constructing toric quantum codes has motivated considerable interest in the literature. This problem was generalized within the context of surface codes [8] and color codes [3]. The most popular toric code was the first proposed by Kitaev's [5]. This code defined on a square lattice of size $m \times m$ on the torus. The parameters of this class of codes are $[[n, k, d]] = [[2m^2, 2, m]]$. In the similar way, the authors in [7] have introduced a construction of topological quantum codes in the projective plane \mathbb{RP}^2 . They showed that the original Shor's 9-qubit repetition code is one of these codes which can be constructed in a planar domain.

Leslie in [6] proposed a new type of sparse CSS quantum error correcting codes based on the homology of hypermaps defined on an $m \times m$ square lattice. The parameters of hypermap-homology codes are $[[\frac{3}{2}m^2, 2, m]]$. These codes are more efficient than Kitaev's toric codes. This seemed to suggest that good quantum codes maybe constructed by using hypergraphs. But there are other surface codes with better parameters than the $[[2m^2, 2, m]]$ toric code. There exist surface codes with parameters $[[m^2 + 1, 2, m]]$, called homological quantum codes. These codes were introduced by Bombin and Martin-Delgado [8].

Authors in [9] presented a new class of toric quantum codes with parameters $[[m^2, 2, m]]$, where $m = 2(l + 1), l \geq 1$. Sarvepalli [10] studied relation between surface codes and hypermap-homology quantum codes. He showed that a canonical hypermap code is identical to a surface code while a noncanonical hypermap code can be transformed to a surface code by CNOT gates alone. Li et al. [17] were given a large number of good binary quantum codes of minimum distances five and six by Steane's Construction. In [18] good binary quantum stabilizer codes are obtained via graphs of Abelian and non-Abelian groups schemes. In [19], Qian presented a new method of constructing quantum codes from cyclic codes over finite ring $F_2 + vF_2$.

Our aim in this work is to present two new classes of binary quantum codes with parameters $[(2k' + 2)(8k' + 7), 2(8k'^2 + 7k'), d_{min}]$ and $[(2k' + 2)(8k' + 9), 2(8k'^2 + 9k' + 1), d_{min}]$ respectively, based on results of Hill in self-complementary self-dual graphs [13]. Binary quantum codes are defined by pair (H_X, H_Z) of \mathbb{Z}_2 -matrices with $H_X H_Z^T = 0$. These codes have parameters $[[n, k, d_{min}]]$, where k logical qubits are encoded into n physical qubits with minimum distance d_{min} . A minimum distance d_{min} code can correct all errors up to $\lfloor \frac{d_{min}-1}{2} \rfloor$ qubits. The code rate for these quantum codes of length $n = (2k' + 2)(8k' + 7)$ and $n = (2k' + 2)(8k' + 9)$ is determined by $\frac{k}{n} = \frac{2(8k'^2 + 7k')}{(2k' + 2)(8k' + 7)}$ and $\frac{k}{n} = \frac{2(8k'^2 + 9k' + 1)}{(2k' + 2)(8k' + 9)}$, and this rate approaches 1 as k' goes to infinity.

The paper is organized as follows. The definition simplices, chain complexes and homology group are recalled in Section 2. In Section 3 we shall briefly present the voltage graphs and their derived graphs. In Section 4, we give a brief outline of self-complementary self-dual graphs. Section 5 is devoted to present new classes of binary quantum codes by using self-complementary self-dual orientable embeddings of voltage graphs and Paley graphs. The paper is ended with a brief conclusion.

2. Homological algebra

In this section, we review some fundamental notions of homology spaces. For more detailed information about homology spaces, refer to [4], [12].

Simplices. Let $m, n \in \mathbb{N}$, $m \geq n$. Let moreover the set of points $\{v_0, v_1, \dots, v_n\}$ of \mathbb{R}^m be geometrically independent. A n -simplex Δ is a subset of \mathbb{R}^m given by

$$\Delta = \{x \in \mathbb{R}^m | x = \sum_{i=0}^n t_i v_i; 0 \leq t_i \leq 1; \sum_{i=0}^n t_i = 1\}. \quad (2.1)$$

Chain complexes. Let K be a simplicial complex and p a dimension. A p -chain is a formal sum of p -simplices in K . The standard notation for this is $c = \sum_i n_i \sigma_i$, where $n_i \in \mathbb{Z}$ and σ_i a p -simplex in K . Let $C_p(K)$ be the set of all p -chains on K . The *boundary homomorphism* $\partial_p : C_p(K) \rightarrow C_{p-1}(K)$ is defined as

$$\partial_k(\sigma) = \sum_{j=0}^k (-1)^j [v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k]. \quad (2.2)$$

The *chain complex* is the sequence of chain groups connected by boundary homomorphisms,

$$\dots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \dots \quad (2.3)$$

Cycles and boundaries. We are interested in two subgroups of $C_p(K)$, *cycle* and *boundary* groups. The p -th cycle group is the kernel of $\partial_p : C_p(K) \rightarrow C_{p-1}(K)$, and denoted as $Z_p = Z_p(K)$. The p -th boundary group is the image of $\partial_{p+1} : C_{p+1}(K) \rightarrow C_p(K)$, and denoted as $B_p = B_p(K)$.

Definition 2.1 (Homology group, Betti number). The p -th *homology group* H_p is the p -th cycle group modulo the p -th boundary group, $H_p = Z_p/B_p$. The p -th *Betti number* is the rank (i.e. the number of generators) of this group, $\beta_p = \text{rank } H_p$. So the first homology group H_1 is given as

$$H_1 = Z_1/B_1. \quad (2.4)$$

From the algebraic topology, we can see that the group H_1 only depends, up to isomorphisms, on the topology of the surface [4]. In fact

$$H_1 \simeq \mathbb{Z}_2^{2g}. \quad (2.5)$$

where g is the genus of the surface, i.e. the number of “holes” or “handles”. We then have

$$|H_1| = 2^{2g}. \quad (2.6)$$

3. Voltage graphs and their derived graphs

Let $G = (V, E)$ be a multigraph for which every edge has been assigned a direction, and \mathcal{V} be a finite group. A *voltage assignment* of G in \mathcal{V} is a function $\alpha : E \rightarrow \mathcal{V}$, that labels the arcs of G with elements of \mathcal{V} . The triple (G, \mathcal{V}, α) is called an (ordinary) voltage graph. The *derived graph* (*lift*, or *covering*) $G' = (V', E')$ (also denoted G^α), is defined as follows:

i) $V' = V \times \mathcal{V}$

ii) If $e = (a, b) \in E$ where $a, b \in V$ and $\alpha(e) = v$ for some $v \in \mathcal{V}$, then $(e, u) = e_u = (a_u, b_{uv}) \in E'$ where $a_u, b_{uv} \in V'$ for all $u \in \mathcal{V}$.

Definition 3.1. Let \mathbb{Z}^t denote $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_t$. A *binary vector* has even weight if it has an even number of 1's and has odd weight otherwise. An ε -vector is a vector in \mathbb{Z}_2^{t-1} with even weight. A σ -vector is a vector in \mathbb{Z}_2^{t-1} with odd weight. We label the ε -vectors so that $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{2^{t-2}}$. Similarly, label the σ -vectors so that $\sigma_1 < \sigma_2 < \cdots < \sigma_{2^{t-2}}$.

Definition 3.2. A 1ε -vector is a vector in \mathbb{Z}_2^t where the first entry is a one and the remainder of the vector is an ε -vector. The 1σ -, 0ε - and 0σ -vectors can be defined in a similar fashion. A 1ε -edge is an edge with a 1ε -vector as a voltage assignment. The 1σ -, 0ε - and 0σ -edges can be defined in a similar fashion. For example, when $t = 3$, we have the following table:

$0\varepsilon_1 = 000 = 0$	$0\sigma_1 = 001 = 1$
$0\varepsilon_2 = 011 = 3$	$0\sigma_2 = 010 = 2$
$1\varepsilon_1 = 100 = 4$	$1\sigma_1 = 101 = 5$
$1\varepsilon_2 = 111 = 7$	$1\sigma_2 = 110 = 6$

Definition 3.3. A *link* is an edge which is incident with 2 different vertices. A *loop* is an edge which has two incidences with the same vertex. A *half edge* is an edge together with one of its incident vertices.

Definition 3.4. Let $t \geq 3$. Let H_t be a voltage graph defined as follows over the group (\mathbb{Z}_2^t, \oplus) ; H_t has two vertices, u and v . There are 2^{t-1} links between u and v , with voltage assignments $0\sigma_1, \dots, 0\sigma_{2^{t-2}}$ and $1\varepsilon_1, \dots, 1\varepsilon_{2^{t-2}}$ (equivalently, all possible vectors in \mathbb{Z}_2^t with odd weight). There are 2^{t-1} half edges about v with voltage assignments $0\sigma_1, \dots, 0\sigma_{2^{t-2}}$ and $1\sigma_1, \dots, 1\sigma_{2^{t-2}}$. Similarly, there are $2^{t-1} - 1$ half edges about u with voltage assignments $0\varepsilon_2, 0\varepsilon_3, \dots, 0\varepsilon_{2^{t-2}}$ and $1\varepsilon_1, \dots, 1\varepsilon_{2^{t-2}}$.

4. Self-complementary self-dual graphs

Let $G = (V, E)$ be a simple graph. The complement \overline{G} of G has the same vertices as G , and every pair of vertices are adjacent by an edge in \overline{G} if and only if they are not adjacent in G . A graph G is self-complementary if $G \cong \overline{G}$. Let $M = (V, E, F)$

be a fixed map of G , with dual map $M^* = (F^*, E^*, V^*)$. M is graphically self-dual if $(V, E) \cong (F^*, E^*)$.

Theorem 4.1. If G is a self-complementary graph on m vertices, then $|E(G)| = \frac{m(m-1)}{4}$, and $m \equiv 0$ or $1 \pmod{4}$.

Proof. See [14].

Theorem 4.2. If G is a self-complementary self-dual graph on m vertices with a self-dual embedding on an orientable surface of genus g , then $m \equiv 0$ or $1 \pmod{8}$. In particular, if $m = 8 + 8k'$, then $g = 8(k'^2) + 7k'$, and if $m = 9 + 8k'$, then $g = 8(k'^2) + 9k' + 1$.

Proof. See [13].

5. Quantum codes from graphs on surfaces

The idea of constructing CSS (Calderbank-Shor-Steane) codes from graphs embedded on surfaces has been discussed in a number of papers. See for detailed descriptions e.g. [11]. Let X be a compact, connected, oriented surface (i.e. 2-manifold) with genus g . A tiling of X is defined to be a cellular embedding of an undirected (simple) graph $G = (V, E)$ in a surface. This embedding defines a set of faces F . Each face is described by the set of edges on its boundary. This tiling of surface is denoted $M = (V, E, F)$. The dual graph G is the graph $G^* = (V^*, E^*)$ such that:

- i) One vertex of G^* inside each face of G ,
- ii) For each edge e of G there is an edge e^* of G^* between the two vertices of G^* corresponding to the two faces of G adjacent to e .

It can be easily seen that, there is a bijection between the edges of G and the edges of G^* .

There is an interesting relationship between the number of elements of a lattice embedded in a surface and its genus. The Euler characteristic of X is defined as its number of vertices ($|V|$) minus its number of edges ($|E|$) plus its number of faces ($|F|$), i.e.,

$$\chi = |V| - |E| + |F|. \quad (5.1)$$

For closed orientable surfaces we have

$$\chi = 2(1 - g). \quad (5.2)$$

The *surface code* associated with a tiling $M = (V, E, F)$ is the CSS code defined by the matrices H_X and H_Z such that $H_X \in \mathcal{M}_{|V|,|E|}(\mathbb{Z}_2)$ is the vertex-edge incidence matrix of the tiling and $H_Z \in \mathcal{M}_{|F|,|E|}(\mathbb{Z}_2)$ is the face-edge incidence matrix of the tiling. Therefore, from (X, G) is constructed a CSS code with parameters $[[n, k, d]]$, where n is the number of edges of G , $k = 2g$ (by (2.6)) and d is the shortest non-boundary cycle in G or G^* . In this work, the minimum distance of quantum codes by a parity check matrix H (or generator matrix) is obtained. For a detailed information to compute the minimum distance, we refer the reader to [15].

5.1. New class of $[[(2k' + 2)(8k' + 7), 2(8k'^2 + 7k'), d_{min}]]$ binary quantum codes from embeddings of voltage graphs

Our aim in this subsection is to construct new class of binary quantum codes by using self-complementary self-dual orientable embeddings of voltage graphs. Let G_t be the lift of voltage graph H_t defined over the group (\mathbb{Z}_2^t, \oplus) . Since $|V(G_t)| = |V(H_t)| \times |\mathbb{Z}_2^t| = 2 \times 2^t$, for $t = 3$, $m = |V(G_t)| = 2^3 \times 2 = 2^4 \equiv 0 \pmod{8}$. On the other hand, since by Theorems in Section 4, $|E(G)| = \frac{m(m-1)}{4}$ and $m = 8 + 8 \times 1$, thus $|E(G)| = 60$ and $g = 15$. From Definition 3.4 we get the following adjacency matrix for $t = 3$:

$$A = \begin{pmatrix} IXX + XII + XXX & XII + IXI + IIX + XXX \\ XII + IXI + IIX + XXX & IIX + IXI + XIX + XXI \end{pmatrix}$$

where I is an 2×2 identity matrix and X is an Pauli matrix. Also, we will sometimes use notation where we omit the tensor signs. For example IXX is shorthand for $I \otimes X \otimes X$. After finding the vertex-edge incidence matrix H_X using the above adjacency matrix and the face-edge incidence matrix H_Z by Gaussian elimination and the *standard form* of the parity check matrix in [15], one can be easily seen that $H_X H_Z^T = 0$ and $d_{min} = 3$. Therefore, the code with parameters $[[60, 30, 3]]$ is constructed.

In general, the adjacency matrix $A = (a_{ij})_{2^{t+1} \times 2^{t+1}}$ of derived voltage graph by Definition 3.4, is

$$A = \begin{pmatrix} B & C \\ C & D \end{pmatrix}$$

where

$$\begin{aligned} B = & \frac{1}{2}(I + X) \otimes \underbrace{\{(I + X) \otimes (I + X) \otimes \cdots \otimes (I + X) + \\ & + \underbrace{(I - X) \otimes (I - X) \otimes \cdots \otimes (I - X)\}_{t-1} - \underbrace{I \otimes I \otimes \cdots \otimes I}_t\}_{t-1}}_{t-1} \end{aligned}$$

$$\begin{aligned}
C &= \frac{1}{2} \{ \underbrace{(I+X) \otimes (I+X) \otimes \cdots \otimes (I+X)}_t - \\
&\quad - \underbrace{(I-X) \otimes (I-X) \otimes \cdots \otimes (I-X)}_t \}; \\
D &= \frac{1}{2} (I+X) \otimes \{ \underbrace{(I+X) \otimes (I+X) \otimes \cdots \otimes (I+X)}_{t-1} - \\
&\quad - \underbrace{(I-X) \otimes (I-X) \otimes \cdots \otimes (I-X)}_{t-1} \}.
\end{aligned}$$

With finding the matrix H_X using the above adjacency matrix $A = (a_{ij})_{2^{t+1} \times 2^{t+1}}$ and the matrix H_Z by Gaussian elimination and the standard form of the parity check matrix, the code minimum distance of at least three is obtained.

After determining d_{min} , by using the Theorems in Section 4 the class of codes with parameters $[[(2k' + 2)(8k' + 7), 2(8k'^2 + 7k'), d_{min}]]$, $k' \geq 1$ is constructed.

5.2. New class of $[[(2k' + 2)(8k' + 9), 2(8k'^2 + 9k' + 1), d_{min}]]$ binary quantum codes from embeddings of Paley graphs

The construction of this class will be based on self-complementary self-dual orientable embeddings of Paley graphs in the Galois field $GF(p^r)$, where $p \in \mathbb{P}$ and $r \in \mathbb{Z}^+$.

Definition 5.2.1. Let G be a group and S be a subset of $G \setminus \{id\}$. We say that a graph X is a *Cayley graph* with *connection set* S , written $X = \text{Cay}(G, S)$, if

- i) $V(X) = G$,
- ii) $E(X) = \{ \{g, sg\} | g \in G, s \in S \}$.

Definition 5.2.2. Let $m = p^r \equiv 1 \pmod{8}$, $p \in \mathbb{P}$ and $r \in \mathbb{Z}^+$. A *Paley graph* is a cayley graph $P_m = \text{Cay}(X_m, \Delta_m)$, where $X_m = \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_m$ is the additive group of the Galois field $GF(p^r)$ and $\Delta_m = \{1, x^2, x^4, \dots, x^{m-3}\}$ for a primitive element x of $GF(p^r)$.

Let $G = (V, E)$ be a self-complementary self-dual graph on m vertices. From Theorem 4.1, we know that $|E(G)| = \frac{m(m-1)}{4}$. Also, from Theorem 4.2 and Definition 5.2.2, with a self-dual embedding on an orientable surface of genus g , we know that if $m = 9 + 8k' \equiv 1 \pmod{8}$, then $g = 8(k'^2) + 9k' + 1$. Therefore,

$|E(G)| = \frac{(9+8k')(8+8k')}{4} = (9+8k')(2+2k')$. Since in this self-dual embedding on an orientable surface the code minimum distance is at least three. Thus the code parameters are given by: the code minimum distance is $d_{min} \geq 3$; the code length is $n = |E(G)| = (9+8k')(2+2k')$ and $k = 2g = 2(8k'^2 + 9k' + 1)$. Consequently, the class of codes with parameters $[[(2k' + 2)(8k' + 9), 2(8k'^2 + 9k' + 1), d_{min}]]$, $k' \geq 0$ is obtained.

Example 5.2.1. Let $m = 3^2 \equiv 1 \pmod{8}$. Then $P_9 = \text{Cay}(X_9, \Delta_9)$, where $X_9 = \mathbb{Z}_3 \times \mathbb{Z}_3$ is the additive group of the Galois field $GF(3^2)$ and $\Delta_9 = \{1, x^2, x^4, x^6\}$ for a primitive element x of $GF(3^2)$. In fact, Δ_9 is the set of all squares in $GF(3^2)$. Let $p(x) \in \mathbb{Z}_3[x]$ be an irreducible polynomial of degree 2. Then the elements of $\mathbb{Z}_3[x]/\langle p(x) \rangle$ will be polynomials of degree 1 or less and there will be $3^2 = 9$ such polynomials. So, in terms of representatives, the elements of $GF(9)$ are $\{ax + b | a, b \in \mathbb{Z}_3\}$. We denote these as:

$$\begin{aligned} g_0 &= 0x + 0 & g_3 &= 1x + 0 & g_6 &= 2x + 0 \\ g_1 &= 0x + 1 & g_4 &= 1x + 1 & g_7 &= 2x + 1 \\ g_2 &= 0x + 2 & g_5 &= 1x + 2 & g_8 &= 2x + 2 \end{aligned}$$

Based on results of Conrad in finite fields [16], the monic irreducible quadratics in $\mathbb{Z}_3[x]$ are $x^2 + 1$, $x^2 + x + 2$ and $x^2 + 2x + 2$. Let $p(x) = x^2 + x + 2$. Then $g_3 = x$ is a generator of the nonzero elements in the field $\mathbb{Z}_3[x]/\langle x^2 + x + 2 \rangle$.

$$g_3 = x = g_3$$

$$g_3^2 = x^2 = -x - 2 = 2x + 1 = g_7$$

$$g_3^3 = x(2x + 1) = 2x^2 + x = 2(-x - 2) + x = -x - 1 = 2x + 2 = g_8$$

$$g_3^4 = x(2x + 2) = 2x^2 + 2x = 2(-x - 2) + 2x = -4 = 2 = g_2$$

$$g_3^5 = x(2) = 2x = g_6$$

$$g_3^6 = x(2x) = 2x^2 = 2(-x - 2) = -2x - 4 = x + 2 = g_5$$

$$g_3^7 = x(x + 2) = x^2 + 2x = -x - 2 + 2x = x - 2 = x + 1 = g_4$$

$$g_3^8 = x(x + 1) = x^2 + x = -x - 2 + x = -2 = 1 = g_1$$

By Definitions in Subsection 5.2, we get the following adjacency matrix for $GF(9)$:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

After finding the matrices H_X and H_Z using the Theorems in Section 4, the code with parameters $[[18, 2, 3]]$ is obtained. Note that the matrix H_Z is given by Gaussian elimination and the standard form of the parity check matrix in [15].

6. Conclusion

We have considered the presentation of two new classes of binary quantum codes by using self-complementary self-dual orientable embeddings of voltage graphs and Paley graphs. These codes is superior to quantum codes presented in other references. We point out the classes $[[(2k' + 2)(8k' + 7), 2(8k'^2 + 7k'), d_{min}(\geq 3)]]$ and $[[(2k' + 2)(8k' + 9), 2(8k'^2 + 9k' + 1), d_{min}(\geq 3)]]$ of quantum codes achieving the best ratio $\frac{k}{n}$.

References

- [1] P. W. Shor, Scheme for reducing decoherence in quantum memory, *Phys. Rev. A*, **2** (1995) 2493-2496.
- [2] A. Calderbank, E. Rains, P. Shor, and N. Sloane, Quantum error correction via codes over GF(4), *IEEE Trans. Inform. Theory*, **44** (1998) 1369-1387.
- [3] H. Bombin, and M. A. Martin-Delgado, Topological quantum distillation, *Phys. Rev. Lett.*, **97** (2006) Article Id. 180501.
- [4] M. Nakahara, Geometry, Topology and Physics, Second Edition, IOP Publishing Ltd, UK, (2003).
- [5] A. Yu. Kitaev, Fault-tolerant quantum computation by anyons, *Annals of Physics*, **303** (2003) 2-30.
- [6] M. Leslie, Hypermap-homology quantum codes, *Int. J. Quantum Inform.*, **12** (2014) p. 1430001.

- [7] M. H. Freedman, D. A. Meyer, Projective plane and planar quantum codes, *Found. Comput. Math.*, **1** (2001) 325-332.
- [8] H. Bombin, and M. A. Martin-Delgado, Homological error correction: Classical and quantum codes, *J. Math. Phys.*, **48** (2007) Article Id. 052105.
- [9] C. D. de Albuquerque, R. P. Junior, and E. B. da Silva, On toric quantum codes, *Int. J. Pure and Applied Math.*, **50** (2009) 221-226.
- [10] P. Sarvepalli, Relation between surface codes and hypermap-homology quantum codes, arXiv: quant-ph/1312.6344v2 (2014).
- [11] G. Zemor, On Cayley graphs, surface codes, and the limits of homological coding for quantum error correction, In *Coding and Cryptology, second international workshop IWCC 2009, LNCS 5557*, Springer, (2009) 259-273.
- [12] H. Edelsbrunner, J. Harer, Computational topology, Duke University, (2008).
- [13] A. B. Hill, Self-dual graphs, M.S. Thesis, Waterloo University, (2002).
- [14] A. T. White, Graphs, Groups, and Surfaces, Amsterdam: North-Holland, (1984).
- [15] M. A. Nielsen, and I. L. Chuang, Quantum computation and quantum information, Cambridge University Press, Cambridge, (2000).
- [16] K. Conrad, Finite fields, Connecticut University, (2013).
- [17] R. Li, X. Li, Binary construction of quantum codes of minimum distances five and six, *Discrete Math.*, **308** (2008) 1603-1611.
- [18] A. Naghipour, M. A. Jafarizadeh, and S. Shahmorad, Quantum stabilizer codes from Abelian and non-Abelian groups association schemes, arXiv: quant-ph/1407.6228v2 (2014).
- [19] J. Qian, Quantum codes from cyclic codes over finite ring $F_2 + vF_2$, *Journal of Information and Computational Science*, 10:6 (2013) 1715-1722.